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Spectra of fluctuations of velocity, kinetic energy, and the dissipation rate in strong turbulence

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Following the ideas of operator product expansion, the velocity $\mathbf{v}$, kinetic energy $K=\frac{1}{2}v^2$, and dissipation rate $\epsilon=\nu_0(\partial v_i/\partial x_j)^2$ are treated as independent dynamical variables, each obeying its own equation of motion. The relations $\overline{\Delta u(\Delta K)^2} \propto r$, $\overline{\Delta u(\Delta \epsilon)^2} \propto r^0$, and $\overline{(\Delta u)^5}\approx r\overline{\Delta\epsilon\Delta K}$ are derived. If velocity scales as $(\Delta v)_{\rm rms} \propto r^{(\gamma/3)-1}$, then simple power counting gives $(\Delta K)_{\rm rms} \propto r^{1-(\gamma/6)}$ and $(\Delta\epsilon)_{\rm rms} \propto 1/\sqrt{\overline{(\Delta v)}_{\rm rms}} \propto r^{(1/2)-(\gamma/6)}$. In the Kolmogorov turbulence ($\gamma=4$) the intermittency exponent $\mu=(\gamma/3)-1=\frac{1}{3}$ and $\overline{(\Delta\epsilon)^2}=O(\mathrm{Re}^{1/4})$. The scaling relation for the ϵ fluctuations is a consequence of cancellation of ultraviolet divergences in the equation of motion for the dissipation rate.

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The Kolmogorov relation for the third-order structure function in decaying homogeneous and isotropic turbulence [1], derived directly from the Navier-Stokes equations, reads

$$S_3=\overline{[u(X+r)-u(X)]^3}=-6\overline{u(x)u(x)u(x+r)}$$
$$=-\tfrac{4}{5}\,\bar\epsilon r+6\nu_0(dS_2/dr), \quad (1)$$

where $u(X)$ is the x component of the velocity field $\mathbf{v}$, r is the displacement in the x direction, and $\bar\epsilon=\nu_0\overline{(\partial v_i/\partial x_j)^2}=O(1)$. The correlation function $S_2=\overline{[u(x)-u(X+r)]^2}$. The mean dissipation rate $\bar\epsilon$ in the Kolmogorov derivation is defined as $\partial\overline{v^2}/\partial t=-2\bar\epsilon$. In a statistically steady flow driven by the force $\mathbf{f}$, $d\overline{v^2}/dt=0=-2\bar\epsilon+2\overline{\mathbf{f}\cdot\mathbf{v}}$ and, in general, the Kolmogorov relation (1) must be modified. The Navier-Stokes equations driven by a force $\mathbf{f}$ are

$$(\partial v_i/\partial t)+v_j(\partial v_i/\partial x_j)=-(\partial p/\partial x_i)+\nu(\partial^2 v_i/\partial x_j^2)+f_i, \quad (2)$$

with $\nabla\cdot\mathbf{v}=0$. It follows from this equation [2] that

$$S_3=-\tfrac{4}{5}\bar\epsilon r+\frac{6}{r^4}\int_0^r y^4\overline{\Delta u\Delta f}\,dy+6\nu\,\frac{\partial S_2}{\partial r}, \quad (3)$$

where $\Delta f=f(X+r)-f(X)$. It is clear that if the energy source acts at the largest scales only, so that the Fourier transform $f(k)=0$ for $k>k_0\to 0$, then the relations (3) and (1) are identical for small enough values of the displacement r. Applying dimensional considerations to the expression (1) or (3) leads to the Kolmogorov law: $(\Delta u)_{\rm rms}=O(\bar\epsilon^{1/3}r^{1/3})$. This relation defines the dimensionality of the velocity operator: the root-mean-square velocity v_r averaged over a ball of a radius r scales as $v_r\propto r^{1/3}$. It is convenient to introduce the effective viscosity $\nu(r)$ defined from the equation for the velocity field $\mathbf{v}^<$, averaged over small-scale ($l<r$) fluctuations: $\mathbf{v}^<\cdot\nabla\mathbf{v}^<\approx\nu(\mathbf{r})\nabla^2\mathbf{v}^<$. Power counting gives $\nu(r)\approx\bar\epsilon^{1/3}r^{4/3}$ in accord with Kolmogorov theory. The effective viscosity $\nu(r)$ takes into account the effects of the velocity fluctuations at scales $l<r$. Dimensional considerations, applied to relation (3), are a very crude approximation. In general, if $(\Delta u)_{\rm rms}\propto r^{(\gamma/3)-1}$, then $\nu(r)\propto r^{\gamma/3}$, where the exponent γ is to be determined from the theory. The Kolmogorov spectrum, approximately supported by experimental data, corresponds to $\gamma=4$. So, in what follows this value of parameter γ will be used for evaluation of the exponents following the general relations derived below. The relation (1) or (3) gives an important constraint on the structure of turbulence theory. Fourier transform of $S_3(k)\propto\delta'(k)=0$ for the wave number k in the inertial range. This means that the largest scales in the flow play the most important part in dynamics of turbulence. To make this point even stronger, let us consider a flow driven by a white-in-time random force having nonzero Fourier component $f(k)$ at $k=k_0$. In this case the relation (3) gives $S_3=-(4\bar\epsilon/5!k_0r^4)(\partial^3/\partial q^3)(\sin qr/q)$ evaluated at $q=k_0$. Here $\bar\epsilon=\langle\mathbf{f}\cdot\mathbf{v}\rangle$. This exact relation tells us that the integral scale k_0 cannot disappear from the problem as a result of Galileo-like transformations. The effects of the large-scale dynamics on the scaling properties of both kinetic energy and dissipation rate will be investigated below.

It is also interesting to investigate the properties of fluctuations of the local values of kinetic energy $K=\frac{1}{2}v_i^2(x)$ and dissipation rate $\epsilon=\nu_0(\partial v_i/\partial x_j)^2$. Power counting based on $u=O(r^{1/3})$ gives for the dimensionality of kinetic energy and dissipation rate: $K^2=O(r^{4/3})$ and $\epsilon_r^2=O(r^{-8/3})$ corresponding to the spectra $E_K(k)\propto k^{-7/3}$ and $E_\epsilon\propto k^{5/3}$. These relations strongly contradict all available experimental data. In this work, following the ideas of operator product expansion, we consider $\mathbf{v}$, K, and ϵ as independent dynamical variables and derive scaling properties of fluctuations of kinetic energy (K) and dissipation rate (ϵ). The equation of motion for kinetic energy is

$$(\partial K/\partial t)+v_j(\partial K/\partial x_j)=-\epsilon-(\partial v_ip/\partial x_i)+\nu_0(\partial^2K/\partial x_j^2)$$
$$+v_if_i. \quad (4)$$

This is essentially an equation for a passive scalar (K) with various sources added to the right side. Repeating the procedure for the isotropic and homogeneous flow, described in Ref. [2] and Monin and Yaglom [3], gives $\overline{\Delta u(\Delta K)^2}=4\overline{u(x)K(x)K(x+r)}\approx-\frac{4}{3}N_Kr+F_K$, where $F_K=O(\overline{\Delta\epsilon\Delta K}r)$ and $N_K=\nu_0\overline{(\partial K/\partial x_j)^2}+4\overline{\epsilon(x)K(x)}=O(1)$. The $O(\Delta u\Delta p\Delta K)$ terms, coming from pressure contributions to (4), are small in the inertial range limit $r\to 0$. It will be shown below that $F_K=O(r^{5/3})$ is also small in the inertial range when displacement r is small enough. This leads to the root mean square of kinetic energy fluctuations:

$\overline{(\Delta K)^2} \propto r^{2-(\gamma/3)} \approx (N_K/\bar{\epsilon}^{1/3}) r^{1/3}$ and $E_K \propto k^{(\gamma/3)-3} \approx k^{-5/3}$ for $\gamma=4$. Again, application of the dimensional reasoning leading to this result is extremely dangerous. Here it is used only to illustrate how the arguments based on the weak coupling, when applied to relevant equations of motion, can lead to the nontrivial scaling. The correlation of kinetic energy fluctuations can also be derived from the following considerations: the Fourier transform of $\overline{(\Delta K)^2}$ involves the integral $\int \langle K(q_1)K(q_2)K(q_3)K(k-q_1-q_2-q_3)\rangle$. If the main contribution to this integral comes from the largest scales, then it is easy to show that

$$\overline{(\Delta K)^2} \approx V_{\rm rms}^2 \overline{(\Delta u)^2} \approx V_{\rm rms}^2 \bar{\epsilon}^{2/3} r^{2/3} (r/L)^{[2(\gamma-1)/3]-2}.$$

It is interesting that these two expressions for the kinetic energy correlation functions coincide only when $\gamma=4$. In what follows we set $\bar{\epsilon}=V_{\rm rms}=L=O(1)$.

The spectrum of the dissipation rate fluctuations has to be calculated from the following equation of motion:

$$(\partial\epsilon/\partial t) + v_i \nabla_i \epsilon = F_\epsilon + \nu_o \nabla_i \nabla_i \epsilon, \tag{5}$$

where

$$F_\epsilon = 2\nu_o(\nabla_j v_i)(\nabla_j f_i) - 2\nu_o(\nabla_j v_i)(\nabla_j v_l)(\nabla_l v_i) - 2\nu_o^2(\nabla_j \nabla_l v_i)^2 - 2\nu_o(\nabla_j v_i)(\nabla_i \nabla_j p). \tag{6}$$

Here, the first term represents P_ϵ, the second T_1, and the third T_2. This equation leads readily to

$$\overline{\Delta u(\Delta\epsilon)^2} = 4\overline{u(x)\epsilon(x)\epsilon(x+r)} \approx -\tfrac{4}{3}N_\epsilon r + \phi_\epsilon, \tag{7}$$

where $N_\epsilon = \overline{F_\epsilon(x)\epsilon(x)} = O(1)$ and $\phi_\epsilon = O(\overline{\Delta F_\epsilon \Delta\epsilon r})$. It will be shown below that $N_\epsilon \approx \bar{\epsilon}^2 \bar{K}/\nu(L)$ where $\nu(L) \approx \bar{\epsilon}^{1/3} L^{4/3}$. Thus all contributions to N_ϵ are $O(1)$ which justifies the estimate $N_\epsilon = O(\mathrm{Re}^0)$. Our goal is to evaluate ϕ_ϵ which can be written as

$$\phi_\epsilon = r\int dk(1-e^{ikr})\overline{F_\epsilon(\mathbf{k})\epsilon(-\mathbf{k})}. \tag{8}$$

Thus the problem is reduced to calculation of

$$Y = \overline{F_\epsilon(\mathbf{k})\epsilon(-\mathbf{k})} = -\nu_0 \int d^3q\, d\Omega\, q_i(q+k)_i \times \overline{v_m(\mathbf{q},\Omega)v_m(\mathbf{k}+\mathbf{q},\omega+\Omega)F_\epsilon(\mathbf{k})}. \tag{9}$$

The procedure leading to calculation of Y in the one-loop approximation was developed in Ref. [4]. The main steps are as follows: First we average the expression F_ϵ over the small-scale velocity fluctuations $v(\mathbf{k})$ with wave numbers $\Lambda < k < \Lambda_0 \approx k_d$. It is assumed that $k \ll \Lambda$. The main result of this averaging is in "dressing" the bare viscosity: everywhere we have to substitute ν_0 by the renormalized value $\nu(\Lambda) \approx \bar{\epsilon}^{1/3}\Lambda^{-4/3}$. Thus the small-scale averaging procedure leads to the expressions (6)–(9) but with $\mathbf{v}^<$ instead of $\mathbf{v}$ and $\nu(\Lambda)$ instead of ν_0. Using this result and the fact that the Reynolds number based on $\nu(\Lambda)$ and $v^<$ is $O(1)$ when Λ is in the inertial range, we derive an estimate:

$$Y \approx \bar{\epsilon}\langle F_\epsilon(\mathbf{k})\rangle. \tag{10}$$

This is clear since the largest contribution to the integral

$$I_Y = \nu(\Lambda)\int d^3q\, d\Omega\, q_i(q+k)_i \times \overline{v_m^<(\mathbf{q},\Omega)v_m^<(\mathbf{k}+\mathbf{q},\omega+\Omega)}$$

taken over the interval $0<q<\Lambda$ comes from the interval $k \ll q \approx \Lambda$. In this approximation in the long-time limit $\omega \to 0$, $I_Y \approx \nu(\Lambda)\int dq\, q^2 E(q) \approx \bar{\epsilon} = O(1)$, which is independent of Λ. Correction to (10), coming from the fluctuating contribution to $\epsilon = \bar{\epsilon} + \delta\epsilon$, will be discussed below. The details of evaluation of $\langle F_\epsilon(\mathbf{k})\rangle$ are given in Ref. [4]. It has been shown that all contributions to $\langle F_\epsilon\rangle$, defined by (6), are ultraviolet divergent, i.e., the integrals depend on the uv cutoff k_d. However, an accurate evaluation of the integrals revealed that all divergent terms in (6) cancel in the one-loop approximation and the resulting expression is independent of the ultraviolet cutoff. It is easy to see from (6) that in the zeroth order the uv divergent corrections to P_ϵ and T_2 in (6) cancel each other, reflecting the fact that the mean rates of production and destruction of ϵ are equal in a statistically steady state [4]. It has also been shown [4] that the divergent terms in P_ϵ, T_1, T_2 and the pressure contributions to (6), appearing in the first order of the iteration procedure, sum up to zero. The first nonvanishing correction to $T_2 \approx \nu^2(\Lambda)\int q^2(\mathbf{k}-\mathbf{q})^2 v_i^<(q)v_i^<(k-q)d^3q\, d\Omega_q$ appears in the second order of the iteration procedure, which uses the Navier-Stokes equations symbolically written as $v^< \approx fG + \frac{1}{2}P(\mathbf{k})G\int v^<(q)v^<(k-q)dq$. This correction is equal to

$$F_\epsilon(\mathbf{k}) \approx \nu^2(\Lambda)\int q^2(\mathbf{k}-\mathbf{q})^2 P(\mathbf{q})P((\mathbf{k}-\mathbf{q}))G(q)G(k-q) \times v^<(Q)v^<(q-Q)v^<(p)v^<(k-q-p)dq\, dp\, dQ, \tag{11}$$

where $P(\mathbf{k}) = O(\mathbf{k})$, $k = (\mathbf{k},\omega)$, and $G(k,\omega) = [-i\omega + \nu(\Lambda)k^2]^{-1}$. Relation (11) represents the operator F_ϵ in terms of $\mathbf{v}^<$. To evaluate expressions (9), (10), and (8) we have to calculate $\bar{F}_\epsilon$. The estimate can be derived readily: iterating (11) using the zeroth-order solution:

$$\overline{v_i(k,\omega)v_j(k',\omega')} \propto \bar{\epsilon}k^{-3}GG^*\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')$$

gives

$$F_\epsilon(\mathbf{k}) \approx \frac{\bar{\epsilon}K(k)}{\nu(\Lambda)}, \tag{12}$$

where

$$K(k) \approx \frac{1}{4}\left[\overline{\left(\int d^3q\, d\Omega_q v_i^<(\mathbf{q},\Omega_q)v_i^<(\mathbf{k}-\mathbf{q},\omega-\Omega_q)\right)^2}\right]^{1/2}.$$

Substituting (10) and (12) into (8) gives for $\Lambda \approx r^{-1}$ and $\omega \to 0$:

$$\overline{\Delta u(\Delta\epsilon)^2} = 4\overline{u(x)\epsilon(x)\epsilon(x+r)} \approx -\tfrac{4}{3}N_\epsilon r + \phi_\epsilon \approx r\frac{\bar{\epsilon}^2(\Delta K)_{\rm rms}}{\nu(r)} = O(\bar{\epsilon}^2 V_{\rm rms} r^0). \tag{13}$$

It is interesting that the scaling of the right side of (13) is independent of the value of γ. This result can be obtained from (11) on the basis of dimensional considerations: the one-loop iteration introduces a factor $\bar{\epsilon}=O(1)$ and $\int G^4 d\Omega$ gives $\nu^3(\Lambda)$ in the denominator. The remaining $O(v^2)$ contributions give ΔK in relation (13). Thus it follows from (7) and (13) that

$$\overline{(\Delta\epsilon)^2}\propto 1/(\Delta u)_{\rm rms}\propto r^{1-(\gamma/3)}\equiv r^{-\mu}\approx r^{-1/3} \quad . \tag{14}$$

This expression defines the so-called intermittency exponents μ. The corresponding spectrum E_ϵ is $E_\epsilon(k) \propto k^{(\gamma/3)-2} \propto k^{-2/3}$. For the mean value of the dissipation rate fluctuations we have $\overline{(\Delta\epsilon)^2} \propto k_d^{1/3} \propto \mathrm{Re}^{1/4}$.

The approach developed in this work can be used for derivation of the scaling properties of various composite operators. For example, the energy equation (4) leads to

$$(\partial/\partial r)\overline{\Delta u(\Delta K)^2}+\tfrac{4}{3}N_K\approx-\overline{\Delta\epsilon\Delta K}. \tag{15}$$

This relation was obtained above taking into account that in the case of the large-scale white-in-time random force $\langle f_i v_i K(x+r)\rangle=\bar{\epsilon}K(x+r)$ and neglecting the pressure contributions. The left side of (15) is $O((\partial/\partial r)[V^2\overline{(\Delta u)^3} + \overline{(\Delta u)^5}]) + \tfrac{4}{3}N_K$, where $V^2 = O((\bar{\epsilon}L)^{2/3}) = O(1)$. Using (1) we come to the conclusion that if the r-independent contributions to the left side of (15), corresponding to the constant in the wave-number space flux of K, cancel each other, the remaining terms give

$$\overline{\Delta\epsilon\Delta K}\approx\frac{\partial}{\partial r}\,\overline{(\Delta u)^5}\propto(r/L)^{\beta}, \tag{16}$$

with $\beta=\tfrac{2}{3}$ if the Kolmogorov prediction for the fifth-order structure function is used. The corresponding co-spectrum $E_{\epsilon,K} \propto k^{-\epsilon/3}$. Using (16) we can show that the contribution to (13), coming from the ϵ fluctuations, neglected in (10), which is $O(r\bar{\epsilon}\;\overline{\Delta\epsilon\Delta K}/\nu(r)\approx r^{1/3})$, is small when $r\to 0$. The relation (16) was verified in numerical experiments on the random force-driven three-dimensional turbulence in Ref. [5]. The theoretical understanding of (16) was developed jointly with the authors of Ref. [5].

All operators $\mathbf{v}$, K, and ϵ, considered in this work are governed by the equations of motion which do not change under Galileo transformations and $K'=K+\tfrac{1}{2}U^2+\mathbf{v}\cdot\mathbf{U}$. It is clear that the scaling of $\overline{(\Delta K')^2}$ with r is the same as that of $\overline{(\Delta K)^2}\approx r^{2/3}$. Indeed, we can write $\overline{(\Delta K')^2}=\overline{(\Delta K)^2}+\tfrac{1}{2}U^2\overline{(\Delta u)^2} \propto r^{2/3}$. The Kolmogorov relation follows directly from the equation for K': $\overline{\Delta u'(\Delta K')^2}=-\tfrac{4}{3}N_{K'}r+F_{K'}$, where $u'=u+U$. Since $\Delta u'=\Delta u$ we have $\overline{(\Delta K')^2} \propto r^{2/3} \propto \overline{(\Delta K)^2}$. As was shown above, the scaling exponent of the right side of relation (13) is invariant under Galileo transformation. However, the proportionality coefficient is transformed as $a'=a+O(U^2)$, i.e., strictly speaking, the relation (13) violates Galileo invariance. This can be an artifact of the low-order diagrammatic approximation used in the derivation of (13) which, in principle, can yield incorrect scaling exponents. So, it is gratifying to know that in the calculation presented here, this is not the case.

Let us explore the possibility that Galileo invariance is broken by powerful large-scale structures, always present in real-life turbulent flows. It is known from experimental data that the large-scale velocity field is described by close-to-Gaussian statistics. To illustrate the physical meaning of the results derived in this work let us write the energy equation:

$$(\partial K/\partial t)+v_j(\partial K/\partial x_j)+V_j(\partial K/\partial x_j)=-\epsilon, \tag{17}$$

where the forcing, pressure, and viscous terms are omitted for simplicity. The large-scale Gaussian velocity field $\mathbf{V}$ is assumed constant in each realization. The nonlinear contribution to (17) can be treated perturbatively. It is easy to see that in the zeroth order:

$$K(k,t)=K(k,0)-\int_0^t\epsilon(k,\tau)e^{i\mathbf{V}\cdot\mathbf{k}\tau}d\tau.$$

Multiplying this equation by $\epsilon(-k,0)$ and averaging independently over the Gaussian field $\mathbf{V}$ and the small-scale ϵ fluctuations we obtain in the long-time limit $t\to\infty$:

$$\overline{\epsilon(-k,0)K(k,0)}\approx\int_0^t\overline{\epsilon(k,\tau)\epsilon(-k,0)}\exp(-\tfrac{1}{4}\,\overline{V^2}k^2\tau^2)d\tau.$$

This integral is evaluated easily when $kV_{\rm rms}\to\infty$:

$$\overline{\epsilon(-k,0)K(k,0)}\approx\frac{1}{kV_{\rm rms}}\,\overline{\epsilon(k,1/kV)\epsilon(-k,0)}$$

$$\approx\frac{1}{kV_{\rm rms}}\,\overline{\epsilon(k,0)\epsilon(-k,0)}. \tag{18}$$

Setting $\tau=(kV_{\rm rms})^{-1}=0$ in the second equality in (18) means that the dynamics of the dissipation rate is characterized by the longer time scale or in other words ϵ, though advected by the large scales, is dominated by the local interactions. This statement will be justified below. Thus the scaling exponent μ of the dissipation rate fluctuations correlation function is given by $-\mu=\beta-1$, where the exponent β is defined by relation (16). This result holds if the neglected nonlinear contribution to the energy equation can be represented in the eddy-viscosity approximation $\nu(k,t)k^2K(k,t)$, provided $kV_{\rm rms}\gg\nu(k,t)k^2$. An attempt to explain experimentally observed large-scale Gaussian statistics of the velocity fluctuations as a result of the symmetry breaking by the large-scale coherent structures was made in Ref. [6].

The relation (18) can be directly obtained from the so-called $K-\epsilon$ model, which is extremely successful in describing large-scale properties of complex turbulent flows. In this model the effective viscosity $\nu\approx(K^<)^2/\epsilon^<$ where the operators $\langle K\rangle_{\rm SS}=K^<$ and $\langle\epsilon\rangle_{\rm SS}=\epsilon^<$ with symbol $\langle\;\rangle_{\rm SS}$ denoting small-scale averaging [4]. The model, which is the result of cancellation of the ultraviolet divergences [4], is

$$(\partial K/\partial t)+v_j(\partial K/\partial x_j)=-\epsilon, \tag{19}$$

$$(\partial\epsilon/\partial t)+v_i\nabla_i\epsilon\approx-C_2(\epsilon^2/K)\;, \tag{20}$$

where, to simplify notation, we set $K=K^<$, $\epsilon^<=\epsilon$ and neglected the diffusion terms in both equations. The constant factor $C_2\approx1.7$ (see Ref. [4]). These equations give readily

$$(\partial/\partial r)\;\overline{\Delta u\Delta K\Delta\epsilon}\approx\overline{(\Delta\epsilon)^2} \tag{21}$$

leading to the relation (18), provided the left side of (21) is dominated by the large-scale advection with velocity $V_{\rm rms}$. The relation (21) is based on the assumption that $\overline{(\epsilon^<)^2}\approx\overline{(\Delta\epsilon)^2}$ if $\epsilon^<$ stands for the dissipation rate field averaged over the scales $l<r$. It is important to notice the difference between (13) and (21). The expression (13), written in terms of Fourier transforms, involves the dissipation rate spectrum $E_\epsilon \propto k^{-2/3}$ and, as a consequence, the integral is infrared convergent, i.e., the resulting expression does not involve both infrared and ultraviolet cutoffs. In this case $(\Delta u)_{\rm rms} \propto r^{1/3}$ is to be used for estimation of the dissipation rate scaling. On the other hand, the relation (21) involves the co-spectrum $E_{\epsilon,K}$ contributing to the strong infrared divergence of the corresponding integral, which leads to the choice $(\Delta u)_{\rm rms}\approx V_{\rm rms}=O(1)$, yielding the same result for the dissipation rate spectrum. It is interesting that the relation (21) is invariant under random Galileo transformation.

The main result of this work is the derivation of the experimentally observable correlation functions:

$$\overline{\Delta u(\Delta K)^2}\approx N_K r, \tag{22}$$

$$\frac{\partial}{\partial r}\overline{\Delta u\Delta\epsilon\Delta K}\approx\overline{(\Delta\epsilon)^2}, \tag{23}$$

$$\overline{\Delta u(\Delta\epsilon)^2}\approx\bar{\epsilon}^2 V_{\rm rms}r^0, \tag{24}$$

$$\overline{(\Delta u)^5}\approx r\overline{\Delta\epsilon\Delta K}, \tag{25}$$

$$r\overline{(\Delta\epsilon)^2}\approx V_{\rm rms}\overline{\Delta\epsilon\Delta K}. \tag{26}$$

Since the effective Reynolds number in the inertial range is $O(1)$ these relations lead approximately to $(\Delta K)_{\rm rms} \propto r^{1/3}$ and $(\Delta\epsilon)_{\rm rms}\propto(\Delta u)_{\rm rms}^{-1/2}\propto r^{-1/6}$. One interesting consequence of relation (13) was noticed by Nelkin [7]: if the Kolmogorov hypothesis $\overline{\epsilon(x)\epsilon(x+r)}\approx\epsilon_r^2\approx(\Delta u)^6/r^2$, where ϵ_r is the value of the dissipation rate ϵ averaged over the sphere of radius r surrounding the point $\mathbf{x}$, is correct, then $\overline{(\Delta u)^7} \propto r^2$. The relation (25) gives also $\overline{(\Delta u)^5} \propto r^{3/2}$. Simple power counting, involved in derivation of these scaling relations, implies weak coupling and thus is a very dangerous procedure. The derived numerical values of the exponents, corresponding to the mean field theory, are not to be taken too seriously despite their close agreement with experimental data.

To conclude: Derivation of the nontrivial spectrum of the dissipation rate fluctuations directly from the Navier-Stokes equations based on a finite order of the renormalized perturbation expansion seems to be impossible. However, even one-loop approximation, applied to the high-order equation of motion for the local values of the dissipation rate, gives a strong intermittency with exponent $\mu=\frac{1}{3}$ in good agreement with experimental data. This result is a direct consequence of exact cancellation of the ultraviolet divergences in the ϵ equation, discovered in Ref. [4] in the context of derivation of the $K-\bar{\epsilon}$ model for the description of the large-scale features of turbulent flows. One consequence of the anomalous scaling of K and ϵ may play an important part in development of turbulence theory: the only dimensionless coupling constant, based on the Kolmogorov scaling, is local Reynolds number $\mathrm{Re}=v_r r/\nu(r)\approx[\bar{\epsilon}r^4/\nu^3(r)]^{1/2}=O(1)$. The nontrivial scaling, derived in this work, leads to appearance of dimensionless parameters $\mathrm{Re}_K \approx K_r r^2/\nu^2(r) \approx \mathrm{Re}^0(r/L)^{-1/3}\rightarrow 0$ and $\mathrm{Re}_\epsilon=\epsilon_r r^4/\nu^3(r)\rightarrow 0$ in the infrared limit $r\rightarrow\infty$. The role of these small parameters in the high-order contributions to the renormalized perturbation expansion is under investigation.

The expression (14) leads to the relation between the shape of the energy spectrum in turbulent flows and the small-scale intermittency exponent μ. Let $E(k) \propto k^{1-(2\gamma/3)}$, so that $u_{\rm rms} \propto r^{(\gamma/3)-1}$. This relation tells us that $\gamma=3$ is a crossover value of parameter γ: the small-scale intermittency exists for $\gamma>3$ and the intermittency exponent $\mu=(\gamma/3)-1$ reaches value $\mu=\frac{1}{3}$ on the Kolmogorov spectrum ($\gamma=4$). It is interesting that at $\gamma>3$, the kinetic energy of the flow is dominated by the large-scale dynamics, while when $\gamma<3$ the main contribution to turbulent energy comes from small scales. This result stresses the dominant role of the large-scale dynamics in the intermittency of the dissipation rate fluctuations.

The method presented in this work can be directly applied to the equation for a passive scalar T, advected by turbulent velocity field. In this case cancellation of uv divergences in the scalar dissipation rate equation should lead to the following correlation functions: $\overline{\Delta u(\Delta K_T)^2}=O(r)$ and $\overline{\Delta u(\Delta N_T)^2}=O(r^0)$ where $K_T=T^2$ and $N_T=(\nabla T)^2$. The power counting gives then $\Delta K_T \propto r^{1/3}$ and $\Delta N_T \propto r^{-1/6}$. These relations are invariant under Galileo transformations. It is interesting that the reported experimentally observed intermittency exponent for the fluctuations of the scalar dissipation rate is $\mu_T\approx 0.35$ which is extremely close to the prediction of this work: $\mu_T\approx\frac{1}{3}$, provided the velocity field is characterized by the Kolmogorov spectrum.

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